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On the Isotropy of Continuized Dislocated Crystals. I. The Isotropic Lattice Distortion

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The geometrical theory of continuous distributions of dislocations traditionally neglects the dependence of a distribution of dislocations on the existence of point defects created by this distribution (e.g., due to intersections of dislocation lines). In this paper the influence of such point defects on metric properties of the continuized dislocated Bravais crystalline structure is assumed to be isotropic. The influence of the point defects on the distribution of dislocations is then modeled by treating dislocations as those located in a conformally flat space. This approach leads (among others) to new results concerning the geometry of glide surfaces.

1. INTRODUCTION

The influence of many dislocations on mechanical properties of a crystalline solid is described in mechanics of continua by means of the so-called geometrical theory of dislocations (e.g., Bilby *et al.*, 1958; Kröner, 1984; Trzęsowski, 1993, 1994). According to this theory, though the existence of many dislocations breaks the long-range order of a crystalline solid, nevertheless its short-range order is remarkably preserved and the dislocated crystalline solid can be locally approximately described as a (macroscopically small) part of an ideal crystal. On the other hand, it is known that the occurrence of many dislocations in a crystalline solid is accompanied by the appearance of point defects, due, e.g., to intersections of the dislocation lines. For example, two intersecting right (or left) screw dislocations produce a line of (self-) interstitials, and if one screw is right and the other left a line of vacancies is formed (Frank and Steeds, 1975). Point defects can appear also at crossover points of edge dislocations or when two parallel dislocation lines join together (Oding, 1961).

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However, it is known also that dislocations have no influence on the local metric properties of the crystal structure of the body (Kröner, 1985). Consequently, the *short-range order* of a dislocated Bravais crystal (with the above-mentioned secondary appearance of point defects) can be described, in a continuous limit defining the so-called *continuized crystal* (Kröner, 1984, 1986; Trzęsowski, 1993), by means of a triple (Φ, G, g) (Trzęsowski, 1994), where $\Phi = (E_a;\ a = 1, 2, 3)$ is a moving (vectorial) frame globally defined on the body (identified with an open connected subset $\mathcal{B}$ of the Euclidean point space E^3), $G \subset SO(3)$ is a group of rotations describing material symmetries of a macroscopically homogeneous crystalline solid, and g, $[g] = \mathrm{cm}^2$, is a metric tensor with respect to which Φ is orthonormal:

$$g(X) = \delta_{ab} E^a(X) \otimes E^b(X) \tag{1}$$

where $X = (X^A)$ is a Lagrange coordinate system on the body $\mathcal{B}$; we will use the so-called *geometric frame references*, i.e., dimensional coordinate systems such that $[X^A] = [dX^A] = \mathrm{cm}$, $[\partial_A = \partial/\partial X^A] = \mathrm{cm}^{-1}$ (in the cgs units system). $\Phi^* = (E^a)$ is the moving coframe dual to Φ:

$$E_a(X) = \underset{a}{e}^A(X)\partial_A, \qquad E^a(X) = \overset{a}{e}_A(X)\, dX^A \tag{2}$$

$$\overset{a}{e}_A(X)\underset{b}{e}^A(X) = \delta^a_b; \qquad [E_a] = \mathrm{cm}^{-1}, \quad [E^a] = \mathrm{cm}$$

The moving frame Φ is (in general) the anholonomic one:

$$[E_a, E_b] = C^c_{ab} E_c \tag{3}$$

where $[u, v] = u \circ v - v \circ u$ denotes the commutator product (bracket) of vector fields u and v considered as first-order differential operators, and smooth scalars C^a_{bc} constitute the so-called object of anholonomity (of Φ).

The vector fields E_a define, at each point of the body (in like manner as in the case of a discrete Bravais crystalline structure), a triple of local crystallographic directions and scales of a locally Euclidean *internal length measurement* along them. It ought to be stressed that the base vectors E_a do not describe translational symmetries of an ideal local lattice (even in the case of a monocrystalline solid). This is because in a continuized crystal translational symmetries are lost and only rotational symmetries (of the considered crystalline material) are preserved (Trzęsowski, 1993). The metric tensor g defining the (non-Euclidean) internal length measurement in the body represents the property of the dislocated crystalline solid that dislocations as well as the secondary point defects created by them have no influence on its local metric properties. The subgroup G [of the group $SO(3)$ of all proper orthogonal matrices 3×3] can be identified with the group of point symmetries of an ideal Bravais reference lattice defining a discrete crystalline structure of the solid in its reference configuration (identified with $\mathcal{B}$), or G can be identified with the group of symmetries of a crystal texture; for isotropic bodies $G = SO(3)$. The pair (Φ, G) is called a *Bravais moving frame*; since G is fixed here, we will identify the Bravais moving frame with the moving frame Φ.

The internal length measurement metric tensor depends on the distribution of point defects (vacancies or interstitials) in the crystal. In particular, each vacancy sits on a crystallographic lattice site, and thus at the junction of three crystallographic lattice lines. Moreover, the frequency of seeing a vacancy along a crystallographic path, measured by lattice steps between neighboring vacancies, is the same in the real and in the continuized crystal. This is due to the requirement that the contents of mass and defect are not changed in the limiting process defining a continuized crystal. Thus, the vacancy, being a crystallographic defect, may be assumed to be isotropic in any Bravais crystal (Kröner, 1990). If interstitials rather than vacancies are considered, then their influence on the internal length measurement can well be anisotropic. This is because the interstitials are never on lattice lines, and thus not in the junction of these lines (Kröner, 1990). However, if we consider, as a particular case or as an approximation, the isotropic influence of interstitials, we can calculate the internal length measurement metric tensor as

$$g_{AB}(Z) \stackrel{*}{=} (1 + \delta(Z))^2 \delta_{AB} \tag{4}$$

where $Z = (Z^A)$ is a Cartesian coordinate system, $\delta > 0$ if the influence of interstitials predominates, $-1 < \delta < 0$ if the influence of vacancies predominates, and $\delta = 0$ if the influence of interstitials and vacancies neutralize each other; $\stackrel{*}{=}$ means that the formula is considered in a (Cartesian) distinguished coordinate system. Let us note that, e.g., in metals in thermal equilibrium the concentration of interstitials may be neglected in comparison with that of vacancies (Hull and Bacon, 1984). On the other hand, usually the ratio of vacancies to atoms in the unit volume is very low, e.g., does not reach more than 10^{-3}, say, in thermal equilibrium (Kröner, 1990). Therefore, one can say that the coefficient δ defining the conformal factor of (4) may be neglected. However, we will see (Section 3) that the existence of this conformal factor influences the geometry of glide surfaces.

The *long-range distortion* of the crystal structure due to dislocations is represented, in the continuized crystal approximation, by means of the so-called *Burgers field* $\tau_\Phi = (\tau^a)$, being a triple of 2-forms defined as

$$\tau^a = dE^a = \tfrac{1}{2}\tau^a_{bc} E^b \wedge E^c, \qquad [\tau^a] = \text{cm} \tag{5}$$

where $\wedge$ denotes the exterior product. It can be shown that (Yano, 1955)

$$\tau^a{}_{bc} = -C^a_{bc} \tag{6}$$

where the object of anholomity C^a_{bc} is defined by (3). If $\tau^a{}_{bc}$ = const, then the Bravais moving frame Φ spans a three-dimensional real Lie algebra $\mathbf{g}_\Phi$ of Φ-parallel vector fields tangent to the body:

$$\mathbf{g}_\Phi = \{\boldsymbol{v} = v^a \boldsymbol{E}_a\colon v^a = \text{const}\} \tag{7}$$

and endowed with the commutation rules defined by (3). The corresponding distribution of dislocations is called *uniformly dense* (Trzęsowski, 1993). It follows from the theory of Lie algebras that there exists [according to the Bianchi classification of these algebras (Barut and Rączka, 1977)] a finite number of types of uniformly dense distributions of dislocations; these are labeled by types of nonisomorphic three-dimensional real Lie algebras (Trzęsowski, 1993). In particular, an Abelian Lie algebra (i.e., with $C^a_{bc} = 0$) describes the case when dislocations are absent.

It follows from (1), (5), and (6) that the internal length measurement metric tensor $\boldsymbol{g}$ associated with an Abelian Lie algebra is flat, and thus we have $\delta = 0$ in (4). This means that this metric tensor depends only on the secondary point defects created by a distribution of dislocations. Note that the flatness of $\boldsymbol{g}$ does not mean that dislocations are absent [see the commentary following (4) and Section 2]. The isotropy condition (4) restricts the considered distributions of dislocations to those for which the internal length measurement metric tensor is conformally flat. Such is, for example, the case of the material Riemannian space $(\mathcal{B}, \boldsymbol{g})$ of a constant scalar (sectional) curvature associated with a uniformly dense distribution of dislocations (Section 3).

We see that the geometrical theory of dislocations should be based on the mutually related triple $(\Phi, \tau_\Phi, \boldsymbol{g})$ of geometric objects rather than (as is usually done) on the pair (Φ, τ_Φ) (e.g., Bilby *et al.*, 1958) and a complementary object (e.g., a Riemannian metric or mass density) representing the possibility of the point defect occurrence in a crystal with dislocations (e.g., Kröner, 1984; Davini and Parry, 1991). In this paper these mutual relations are modeled by treating dislocations as those located in a conformally flat space (Section 2). We obtain, then, e.g., a generalization (taking into account the influence of point defects) of the well-known Bilby *et al.* (1958) results concerning the geometry of glide surfaces, and a generalization of a formula (Orlov, 1983) describing the influence of dislocations on the mean curvature of a crystalline network (Section 3).

2. DENSITY OF DISLOCATIONS

The long-range distortion of a crystal structure due to the influence of many dislocations can be quantitatively measured by the Burgers vector (e.g., Trzęsowski, 1994). Its local counterpart, called a *local Burgers vector*, can be introduced in the following way. If the 2-form $\Sigma^{ab} = E^a \wedge E^b$ is considered as the one representing a surface element $d\Sigma$ of a surface $\Sigma \subset \mathcal{B}$ [treated as a two-dimensional submanifold of the material Riemannian space $(\mathcal{B}, \boldsymbol{g})$; see Section 1] with its unit normal $\boldsymbol{l} = l^a \boldsymbol{E}_a$, i.e., if we identify

$$
\begin{aligned}
d\Sigma_a &= \tfrac{1}{2} e_{abc} \Sigma^{bc} = l_a \, d\Sigma, \qquad l_a = \delta_{ab} l^b \\
e^{abc} &= g^{-1/2} \epsilon^{abc}, \qquad g = \det(g_{ab}) \doteq 1 \\
[l_a] &= [1], \qquad [d\Sigma_a] = [d\Sigma] = \mathrm{cm}^2
\end{aligned}
\tag{8}
$$

where ϵ^{abc} denotes the permutation symbol, then it follows from (5) and (8) that a component τ^a of the Burgers field τ_Φ (Section 1) can be identified with an infinitesimal quantity of the form

$$
\tau^a = \rho \beta^a d\Sigma, \qquad [\rho] = \mathrm{cm}^{-2}, \quad [\beta^a] = \mathrm{cm} \tag{9}
$$

where ρ is a positive scalar independent of the choice of $\boldsymbol{l}$, and we have denoted

$$
\rho \beta^a = l_b \alpha^{ba}, \qquad \alpha^{ab} = \tfrac{1}{2} e^{acd} \tau^b{}_{cd}, \qquad l_a l^a = 1 \tag{10}
$$

Consequently

$$
\begin{aligned}
\tau^a{}_{bc} &= e_{bcd} \gamma^{da} - t_{[b} \delta^a_{c]} \\
\gamma^{ab} &= \alpha^{(ab)}, \qquad t_a = \tau^b{}_{ba} = e_{abc} \alpha^{bc} \\
e_{abc} &= g^{1/2} \epsilon_{abc} \doteq \epsilon_{abc}, \qquad [\gamma^{ab}] = [t_a] = \mathrm{cm}^{-1}
\end{aligned}
\tag{11}
$$

where $\epsilon_{abc} = \epsilon^{abc}$. Interpreting $\boldsymbol{l}$ as the unit vector field tangent to a dislocation line [defined, in the continuous description, as the boundary between slipped and unslipped parts of the crystal (Hull and Bacon, 1984)] normal to the surface element $d\Sigma$, and assuming the scalar ρ to be the (volume) *scalar density of dislocations* defined as the length of all dislocation lines included in the volume unit of the Riemannian space, i.e., with the volume element $dV(X)$ defined as

$$
\begin{aligned}
dV(X) &= g(X)^{1/2} dV_0(X) \\
g(X) &= \det(g_{AB}(X))
\end{aligned}
\tag{12}
$$

where $dV_0(X)$ $(= dX^1 \wedge dX^2 \wedge dX^3)$ covers, in the Cartesian coordinate system of (4), with the Euclidean volume element, we can define the local Burgers vector as $\boldsymbol{\beta} = \beta^a \boldsymbol{E}_a$, $[\boldsymbol{\beta}] = [1]$ (Trzęsowski, 1994). The tensor field $\boldsymbol{\alpha} = \alpha^{ab} \boldsymbol{E}_a \otimes \boldsymbol{E}_b$ is called the *dislocation density tensor.*

A line can be interpreted as an *edge* dislocation line if

$$
\beta^a l_a = 0, \qquad \boldsymbol{\beta} \neq \boldsymbol{0} \tag{13}
$$

or a screw dislocation line if

$$
\beta^a = \eta l^a \tag{14}
$$

In other cases the line is interpreted as a *mixed* (edge and screw) dislocation line. The plane $\pi(\boldsymbol{l}, \boldsymbol{\beta})$ containing vectors $\boldsymbol{l}$ and $\boldsymbol{\beta}$ can be interpreted as a *local slip plane* (Trzęsowski, 1994).

It can be shown that for uniformly dense distributions of dislocations (Section 1) the following conditions are fulfilled [see (11)]:

$$\gamma^{ab} t_b = 0 \tag{15}$$

and

$$dt = 0 \tag{16}$$

i.e., at least locally

$$t = d\varphi, \qquad [\varphi] = [1] \tag{17}$$

and the surfaces $\varphi = \text{const}$ can be considered as (local) *slip surfaces* (Trzęsowski, 1994). If the first the de Rham cohomology class of the three-dimensional manifold $\mathcal{B}$ vanishes (e.g., it is the case of a three-dimensional affine space) or, more generally, if the manifold is contractible to a point, then the potential φ of (17) is defined globally. Since for uniformly dense distributions of dislocations $\gamma^{ab} = \text{const}$ and $t_a = \text{const}$, (15) means that, up to a global rotation of the base fields $\boldsymbol{E}_a$, we can assume that

$$\gamma = \gamma^{ab} \boldsymbol{E}_a \otimes \boldsymbol{E}_b = \gamma^a \boldsymbol{E}_a \otimes \boldsymbol{E}_a \tag{18}$$
$$t = t_a E^a = sE^1, \qquad \gamma^1 s = 0; \qquad [s] = [\gamma^a] = \text{cm}^{-1}$$

and so, in this base,

$$(\alpha^{ab}) = \begin{pmatrix} \gamma^1 & 0 & 0 \\ 0 & \gamma^2 & \mu \\ 0 & -\mu & \gamma^3 \end{pmatrix}, \qquad \mu = s/2, \qquad \mu\gamma^1 = 0 \tag{19}$$

Let us consider the Lie algebra $\mathbf{g}_\Phi$ associated with a uniformly dense distribution of dislocations and defined by (3) and (7) with $C^a_{bc} = \text{const}$. If the corresponding internal length measurement metric tensor $\boldsymbol{g}$ defines a Riemannian space of the constant scalar (sectional) curvature K, then it is an Einstein space with [see (1) and (2)]

$$R_{AB}[\mathbf{g}] = 2K g_{AB} \tag{20}$$
$$g_{AB} = \overset{a}{e}_A \overset{b}{e}_B \delta_{ab}, \qquad [g_{AB}] = [1], \qquad [K] = \text{cm}^{-2}$$

where $R_{AB}[\mathbf{g}]$ denotes the Ricci tensor of $\boldsymbol{g}$, and thus $\boldsymbol{g}$ is conformally flat (Gołąb, 1966). Therefore, in this case the isotropy condition (4) is fulfilled and the Ricci tensor is an isotropic tensor, too. It can be shown, correcting slightly computations of Trzęsowski (1994) and taking into account (11), (15), and (20), that the potential φ of (17) should satisfy the equation

$$\Delta_g\varphi = \tfrac{1}{3}[2\gamma^{ab}\gamma_{ab} - (\gamma^c{}_c)^2] + 4K \tag{21}$$

where Δ_g denotes the Laplace–Beltrami operator on $(\mathcal{B}, \boldsymbol{g})$:

$$\Delta_g\varphi = g^{-1/2}\partial_A(g^{1/2}g^{AB}\partial_B\varphi) \tag{22}$$

and we have denoted

$$\gamma^a{}_b = \delta_{bc}\gamma^{ac}, \qquad \gamma_{ab} = \delta_{ac}\gamma^c{}_b \tag{23}$$

It can be shown also [changing slightly the base fields of Lie algebras considered by Fagundes (1991)] that the only real three-dimensional Lie algebras defining the internal length measurement of a constant scalar curvature K are those of the following types.

(a) $\mathbf{g}_\Phi$ *Abelian* defined by

$$[\boldsymbol{E}_a, \boldsymbol{E}_b] = 0, \qquad \text{i.e., } dE^a = 0 \tag{24}$$

with

$$\gamma^{ab} = 0, \qquad t_a = 0, \qquad K = 0 \tag{25}$$

(b) $\mathbf{g}_\Phi = \mathbf{g}_\kappa$, $0 \leq \kappa \leq 1$, defined by

$$[\boldsymbol{E}_1, \boldsymbol{E}_2] = 2k(\kappa\boldsymbol{E}_2 + \boldsymbol{E}_3), \qquad [\boldsymbol{E}_1, \boldsymbol{E}_3] = 2k(\kappa\boldsymbol{E}_3 - \boldsymbol{E}_2), \qquad [\boldsymbol{E}_2, \boldsymbol{E}_3] = 0$$
$$dE^1 = 0, \qquad k = \text{const} > 0, \qquad [k] = \text{cm}^{-1}, \qquad [\kappa] = [1] \tag{26}$$

with

$$(\gamma^{ab}) = -2k\,\text{diag}(0, 1, 1), \qquad t_a = 4k\kappa\delta^1_a \tag{27}$$

For example, for the base covectors of the form

$$E^1 = du^1, \qquad E^2 = \exp(-k\kappa u^1)(\cos ku^1\, du^2 + \sin ku^1\, du^3) \tag{28}$$
$$E^3 = \exp(-k\kappa u^1)(-\sin ku^1\, du^2 + \cos ku^1\, du^3), \qquad [u^A] = \text{cm}$$

we have

$$dE^1 = 0, \qquad dE^2 = -k(\kappa E^1 \wedge E^2 - E^1 \wedge E^3) \tag{29}$$
$$dE^3 = -k(\kappa E^1 \wedge E^3 + E^1 \wedge E^2)$$

and so, according to (3), (5), and (6), the commutation rules (26) are fulfilled. Then

$$\boldsymbol{g}(u) = du^1 \otimes du^1 + \exp(-2k\kappa u^1)(du^2 \otimes du^2 + du^3 \otimes du^3) \tag{30}$$

and (Yano, 1955)

$$K = K_\kappa = -\kappa/l_d^2 \leq 0, \qquad l_d = 1/k \tag{31}$$

(c) $\mathbf{g}_\Phi = \mathbf{g}_t$ defined by

$$[\boldsymbol{E}_1, \boldsymbol{E}_2] = 2k\boldsymbol{E}_2, \qquad [\boldsymbol{E}_1, \boldsymbol{E}_3] = 2k\boldsymbol{E}_3, \qquad [\boldsymbol{E}_2, \boldsymbol{E}_3] = 0 \tag{32}$$
$$dE^1 = 0, \qquad k = \text{const} > 0, \qquad [k] = \text{cm}^{-1}$$

with

$$\gamma^{ab} = 0, \qquad t_a = 4k\delta_a^1 \tag{33}$$

For example, for base covectors of the form

$$E^1 = du^1, \qquad E^\alpha = \exp(-ku^1)\, du^\alpha \qquad (\alpha = 2, 3) \tag{34}$$

we have

$$dE^1 = 0, \qquad dE^2 = -kE^1 \wedge E^2, \qquad dE^3 = -kE^1 \wedge E^3 \tag{35}$$

and the commutation rules (32) are fulfilled. Since

$$\boldsymbol{g}(u) = du^1 \otimes du^1 + \exp(-2ku^1)(du^2 \otimes du^2 + du^3 \otimes du^3) \tag{36}$$

we obtain that

$$K = -1/l_d^2 < 0 \tag{37}$$

(d) $\mathbf{g}_\Phi = \mathbf{g}_\gamma \simeq so(3)$ defined by

$$[\boldsymbol{E}_a, \boldsymbol{E}_b] = 2k\epsilon_a{}^c{}_b\boldsymbol{E}_c \tag{38}$$
$$\epsilon_a{}^c{}_b = \delta^{dc}\epsilon_{adb}, \qquad k = \text{const} > 0, \qquad [k] = \text{cm}^{-1}$$

with

$$\gamma^{ab} = -2k\delta^{ab}, \qquad t_a = 0 \tag{39}$$

It follows from (21) with $\varphi = \text{const}$ and from (39) that

$$K = 1/l_d^2 > 0 \tag{40}$$

Inserting (27) or (33) into (21), we obtain that for $K < 0$ the following equation should be fulfilled:

$$\Delta_g\varphi = 4K \tag{41}$$

Moreover, for the Lie algebra $\mathbf{g}_0$ defined by [see (26)]

$$[\boldsymbol{E}_1, \boldsymbol{E}_2] = 2k\boldsymbol{E}_3, \qquad [\boldsymbol{E}_1, \boldsymbol{E}_3] = -2k\boldsymbol{E}_2, \qquad [\boldsymbol{E}_2, \boldsymbol{E}_3] = 0 \tag{42}$$

with [see (27)]

$$(\gamma^{ab}) = -2k\,\mathrm{diag}(0, 1, 1), \qquad t_a = 0 \tag{43}$$

we have $K = K_0 = 0$, and (41) is fulfilled by its particular solution $\varphi =$ const [see (17) with $t = 0$]. Note that this Lie algebra is isomorphic to the Lie algebra $e(2)$ of the group $E(2)$ of motions of the plane R^2 (Barut and Rączka, 1977).

From (26) we obtain that structure constants of the Lie algebra $\mathbf{g}_\kappa$ have the form

$$C^c_{ab} = 4k(\kappa\delta^1_{[a}\delta^c_{b]} - \delta^2_{[a}\delta^3_{b]}\delta^c_1 + \tfrac{1}{2}\epsilon_{ab}{}^c) \tag{44}$$

Thus [see (3), (6), (10), and (11)]

$$\rho\beta^a = 2k(\kappa l_b\epsilon^{1ba} + l_1\delta^a_1 - l^a) \tag{45}$$

$$\rho\beta_g = 2k(1 - l_1^2)\sqrt{1 + \kappa^2}, \qquad \beta_g^2 = \beta^a\beta_a$$

It follows from (13) and (14) that there exist then no edge dislocation lines, and for $\kappa \neq 0$ there exist no screw dislocation lines. So, for $\kappa \neq 0$ there exist mixed dislocation lines only. For $\kappa = 0$, lines with $l_1 = 0$ are screw. Since [see (27) and (45)]

$$t_a l^a = 4k\kappa l^1, \qquad \beta^a t_a = 0 \tag{46}$$

surfaces $\varphi =$ const normal to the $\boldsymbol{t}$ $(= 4k\kappa \boldsymbol{E}_1)$ direction are slip surfaces for all curves on these surfaces (i.e., if $l_1 = 0$). Along these slip surfaces we have

$$\rho\beta_g = 2k\sqrt{1 + \kappa^2} \tag{47}$$

From (31) we obtain that for the Lie algebra $\mathbf{g}_t$

$$C^c_{ab} = 4k\delta^1_{[a}\delta^c_{b]} \tag{48}$$

and thus

$$\rho\beta^a = 2k\epsilon^{1ba}l_b, \qquad \rho\beta_g = 2k\sqrt{1 - l_1^2} \tag{49}$$

$$t_a l^a = 4kl^1, \qquad \beta^a t_a = 0, \qquad \beta^a l_a = 0$$

So the Lie algebra $\mathbf{g}_t$ describes edge (and only edge) dislocation lines, surfaces $\varphi =$ const normal to the $\boldsymbol{t}$ $(= 4k\boldsymbol{E}_1)$ direction are slip surfaces for all curves on these surfaces, and along these surfaces

$$\rho\beta_g = 2k \tag{50}$$

For the Lie algebra $\mathbf{g}_\gamma$ [see (38)] we have

$$C^c_{ab} = -2k\epsilon_{ab}{}^c \tag{51}$$

$$\rho\beta^a = -2kl^a, \qquad \rho\beta_g = 2k$$

which means that this Lie algebra describes screw (and only screw) dislocation lines, and there exist no distinguished slip surfaces in this case.

Lattice lines in a continuously dislocated crystal form a system of three independent congruences of curves—trajectories of base vectors of the Bravais moving frame $\Phi = (\boldsymbol{E}_a)$ (Bilby *et al.*, 1958; Trzęsowski, 1994). In general none of these congruences is normal (that is, the curves of the congruence are not the orthogonal trajectories of a family of surfaces). But in the case of the so-called *single glide* (in which the dislocation moves in the surface which contains both the line and Burgers vector), the lattice lines originally normal to the plane of slip do form a normal congruence (Bilby *et al.*, 1958). The surfaces are called *glide surfaces*. It follows from (46) and (49) that slip surfaces φ = const can be considered, for all dislocation lines located on them, as glide surfaces. Moreover, along these glide surfaces, the relation (47) (for $\mathbf{g}_\Phi = \mathbf{g}_\kappa$) or (50) (for $\mathbf{g}_\Phi = \mathbf{g}_t$) is valid.

3. INTERNAL LENGTH MEASUREMENT

It was pointed out in Section 2 that the internal length measurement metric tensor $\boldsymbol{g}$ of a constant (sectional) scalar curvature satisfies the isotropy condition (4). Moreover, there exists then a coordinate system $X = (X^A)$ such that (Eisenhart, 1964)

$$\begin{gathered} g_{AB}(X) = \alpha(x(X))^{-2}\delta_{AB} \\ \alpha(x) = 1 + \tfrac{1}{4}(\operatorname{sgn} K)x^2 > 0, \qquad x(X) = r/r_d \\ r^2 = \delta_{AB}X^AX^B, \qquad r_d = |K|^{-1/2}, \qquad [r_d] = \text{cm} \end{gathered} \tag{52}$$

where K denotes the scalar curvature. The coordinate system of (52) is not, in general, that (Cartesian) of (4). Let us designate by $X^A = X^A(Z)$ the coordinate transformation $Z \to X$ and by $J(Z) > 0$ the Jacobian of this transformation. Computing representations of the Riemannian volume element (12) in both coordinate systems and taking into account that g of (12) is a scalar density of weight +2 (Gołąb, 1966), we obtain that

$$\begin{gathered} \delta(Z) = \frac{J(Z)^{1/3}}{\alpha(x(Z))} - 1, \qquad J(Z) > 0 \\ x(Z) = r(Z)/r_d, \qquad r(Z)^2 = \delta_{AB}X^A(Z)X^B(Z) \end{gathered} \tag{53}$$

In particular, if $J(Z) = 1$, then

$$\delta(Z) = -\operatorname{sgn} K \frac{x(Z)^2}{4\alpha(x(Z))}, \qquad |\delta(Z)| < 1 \tag{54}$$

and from (4) and (54) we conclude the following. If $K > 0$, then $\delta(Z) < 0$ and the influence of vacancies (on the internal length measurement) predominates. If $K < 0$, then $\delta(Z) > 0$ and the influence of interstitials predominates. If $K = 0$, then the influences of vacancies and interstitials neutralize each other (e.g., this is the case of the Lie algebra $\mathbf{g}_0$; Section 2) or point defects are absent (since dislocations are absent). If, additionally, $X(0) = 0$, then $\delta(0) = 0$ and there exist in the neighborhood of $Z = 0$ vacancies as well as interstitials. Note that if $K < 0$, then assuming the existence of a coordinate system $X = (X^A)$ of (52) such that

$$\begin{aligned} &\alpha(x_0) \leq J(Z)^{1/3} \leq \alpha(x(Z)), \qquad x(0) = 0 \\ &\alpha(x) = 1 - \tfrac{1}{4}x^2 > 0, \qquad 0 \leq x \leq x_0 < 2 \end{aligned} \tag{55}$$

we can consider the case when the influence of vacancies predominates in the interior of the sphere $K(0, x_0 r_d) \subset K(0, 2r_d)$.

If the distribution of dislocations is uniformly dense and $K \leq 0$, then (30) and (36) mean that the internal length measurement metric tensor is represented in the so-called geodesic form (Yano, 1955):

$$\begin{aligned} \boldsymbol{g} &= du^1 \otimes du^1 + \Psi(u^1)\boldsymbol{a} \\ \boldsymbol{a} &= \delta_{\alpha\beta} du^\alpha \otimes du^\beta, \qquad \alpha, \beta = 2, 3 \\ \Psi(u^1) &= \exp(-2k\kappa u^1) \qquad \text{for} \quad \mathbf{g}_\Phi = \mathbf{g}_\kappa \\ \Psi(u^1) &= \exp(-2ku^1) \qquad \text{for} \quad \mathbf{g}_\Phi = \mathbf{g}_t \end{aligned} \tag{56}$$

The potential φ of (17) takes, in the coordinate system $u = (u^A)$ of (56), the following form:

$$\begin{aligned} \varphi &= 2Hu^1 \\ H &= 2\kappa k \qquad \text{for} \quad \mathbf{g}_\Phi = \mathbf{g}_\kappa \\ H &= 2k \qquad \text{for} \quad \mathbf{g}_\Phi = \mathbf{g}_t \end{aligned} \tag{57}$$

and the surfaces $\Sigma_c = \{u\colon u^1 = c = \text{const}\}$ normal to the $\boldsymbol{E}_1$ $(= \partial/\partial u^1)$ direction can be considered as glide surfaces along which the relation (47) or (50) is valid (Section 2).

The surfaces Σ_c are two-dimensional submanifolds of the Riemannian material space $(\mathcal{B}, \boldsymbol{g})$ with their first fundamental forms $\boldsymbol{a}_c$ defined by

$$\boldsymbol{a}_c = \Psi(c)\boldsymbol{a} = \underset{c}{a}_{\alpha\beta} du^\alpha \otimes du^\beta, \qquad \underset{c}{a}_{\alpha\beta} = \Psi(c)\delta_{\alpha\beta} \tag{58}$$

The second fundamental form $\boldsymbol{b}_c$ of Σ_c is given by (Eisenhart, 1964)

$$\boldsymbol{b}_c = \underset{c}{b}_{\alpha\beta} du^\alpha \otimes du^\beta, \qquad \underset{c}{b}_{\alpha\beta} = \frac{H}{2}\Psi(c)\delta_{\alpha\beta} \tag{59}$$

where H is the constant defined in (57). This means that Σ_c are the so-called *umbilical surfaces* (i.e., surfaces with indeterminate lines of curvature), being a generalization of a plane or sphere in a Euclidean 3-space. The *mean curvature* H_c of the surface Σ_c (for the normal direction E_1) is a constant independent of c, and

$$H_c = \underset{c}{a}^{\alpha\beta}\underset{c}{b}_{\alpha\beta} = H \tag{60}$$

Since $(\mathcal{B}, g)$ is a space of a constant scalar curvature K [given by (31) for $\mathbf{g}_\Phi = \mathbf{g}_\kappa$ or (37) for $\mathbf{g}_\Phi = \mathbf{g}_t$] and the Σ_c are umbilical surfaces, a surface Σ_c has the constant curvature K_c and (Eisenhart, 1964)

$$K_c = \tfrac{1}{4}H_c^2 + K = 0 \tag{61}$$

where (31), (37), (57), and (60) were taken into account. Thus, the surfaces Σ_c are isometric to the plane, in the small at least. Note that a surface in a three-dimensional flat space on which the scalar curvature vanishes is called a *developable surface*; here it is the case described by the Lie algebra $\mathbf{g}_0$ [(42) and (43)] isomorphic to the Lie algebra $e(2)$ of the group $E(2)$ of motions of the plane R^2. These results concerning the geometry of slip surfaces generalize the well-known (Bilby *et al.*, 1958) results concerning developable glide surfaces. Moreover, the Euclidean group $E(2)$ is represented as a subgroup of the group of motions in the Riemannian manifold $(\mathcal{B}, g)$ (Yano, 1955), and slip surfaces are invariant manifolds of the group $E(2)$ action. This means that the considered Riemannian material space admits as its motion, in the small at least, the deformation of an ideal crystal lattice characterizing the influence of a single glide on this lattice (cf. Bilby *et al.*, 1958).

The relations (47) and (50) can be written in terms of (57) and (60) as

$$\begin{aligned} &\rho(X)\beta_g(X) = \nu H \\ &\nu = \frac{1}{\kappa}\sqrt{\kappa^2 + 1} \qquad \text{for} \quad \mathbf{g}_\Phi = \mathbf{g}_\kappa, \quad 0 < \kappa \le 1 \\ &\nu = 1 \qquad\qquad\quad \text{for} \quad \mathbf{g}_\Phi = \mathbf{g}_t \end{aligned} \tag{62}$$

Since trajectories of a Bravais moving frame are lattice lines of the continuized Bravais crystal (see final remarks of Section 2), the relation (62) generalizes the well-known relation describing the influence of dislocations on the mean curvature of a crystalline network (here the one located on a surface Σ_c) (Orlov, 1983). For $\kappa = 0$ we have $K = 0$ [see (31)], and (47) takes the form (50) with g being a flat metric.

4. EDGE DISLOCATION LOOPS

Let us return to the identification (9) of the Burgers field $\tau_\Phi = (\tau^a)$. It follows from (6), (10), (33), and (48) that the dislocation density tensor of the uniformly dense distribution of edge dislocations defined by the Lie algebra $\mathbf{g}_t$ (Section 2) has the form

$$\alpha^{ab} = \tfrac{1}{2}t_c e^{cab} \tag{63}$$

and (9) can be written as

$$\tau^a = \omega^a d\Sigma \tag{64}$$

$$\omega^a = e^{abc}\Omega_{bc}, \qquad \Omega_{bc} = S_{[b}l_{c]}, \qquad S_c = t_c/2$$

Consequently, the Burgers field τ_Φ can be identified with an infinitesimal vectorial quantity $\boldsymbol{\tau}_\Phi$ of the form

$$\boldsymbol{\tau}_\Phi = \tau^a \mathbf{E}_a = \boldsymbol{\omega}\, d\Sigma, \qquad \boldsymbol{\omega} = \omega^a \mathbf{E}_a \tag{65}$$

where the vector field $\boldsymbol{\omega}$ can be interpreted as a vortex field of vortices in a cylinder tube with the section $d\Sigma$. The tensorial density Ω_{ab} of this vortices is given by

$$\Omega_{ab} = \tfrac{1}{2}e_{abc}\omega^c \tag{66}$$

Comparing (9) and (64), we obtain that

$$\omega^a = \rho\beta^a \tag{67}$$

and (66) becomes

$$\Omega_{ab} = \rho\beta_{ab}, \qquad \beta_{ab} = \tfrac{1}{2}e_{abc}\beta^c \tag{68}$$

Thus, the Lie algebra $\mathbf{g}_t$ can be interpreted as the one describing a system of infinitesimal oriented edge dislocation loops of the scalar density ρ and with the tensorial density β_{ab} of their local Burgers vectors. For example, the irradiation of a crystal with fast neutrons produces very small circular edge dislocation loops (Bullough and Newman, 1970). The loops can be treated then (in the continuized crystal approximation) as being infinitesimal.

Since the glide direction of a loop is parallel to the Burgers vector of the loop, we can consider the local Burgers vector $\boldsymbol{\beta}$ as the one defining a local generalized translation according to (Yano, 1955)

$$L_{\boldsymbol{\beta}}\mathbf{E}_a = 0, \qquad a = 1, 2, 3 \tag{69}$$

where L denotes the Lie derivative operator. It can be shown that if the local Burgers vector $\boldsymbol{\beta}$ corresponds to a Φ-parallel dislocation line (i.e., $\mathbf{l} = l^a\mathbf{E}_a$, $l^a = \text{const}$), then the condition (69) reduces to (Trzęsowski, 1994)

$$\rho = \rho_0 \exp(\varphi/2) \tag{70}$$

where φ is the potential of (17) defined by (41) with $K = -l_d^{-2}$, $l_d = 1/k$ [see (32)–(37)], ρ is the scalar density of dislocation loops, and ρ_0, $[\rho_0] = \mathrm{cm}^{-2}$, is a positive constant. The module β_g of local Burgers vectors of dislocation loops gliding along the surface $\varphi = \mathrm{const}$ can be computed from (62) with $\nu = 1$ and (70). Equations (17), (41), (63), and (70) constitute a static counterpart of the dislocation fluid (of the density ρ) model consisting of infinitesimal edge dislocation loops, and describing (in its dynamical version) the influence of mobile dislocation loops on the crystal lattice plastic isotropic distortion (Trzęsowski, 1989, 1995). If additionally (52) (with sign $K = -1$ and $r_d = l_d$) is taken into account, then we obtain a local [in the interior of the sphere $K(0, 2l_d)$] complete description of the distribution of dislocations.

5. SUMMARY OF THE RESULTS

It was shown in Section 2 that uniformly dense distributions of dislocations consistent with an isotropic internal length measurement (Section 1) are associated with three types of nonisomorphic Lie algebras: the $\mathbf{g}_t$ type describing a distribution of edge dislocations, the $\mathbf{g}_\gamma$ type describing a distribution of screw dislocations, and the $\mathbf{g}_\kappa$-type, $0 \leq \kappa \leq 1$, describing for $\kappa \neq 0$ a distribution of mixed dislocations (for $\kappa = 0$ screw dislocation lines are additionally admitted).

For a distribution of dislocations of $\mathbf{g}_t$ or $\mathbf{g}_\kappa$ types, $\kappa \neq 0$, there exists a geometrically and physically distinguished family of slip surfaces, and an equation defining these surfaces can be formulated (Section 2). It was shown in Section 3 that slip surfaces are then the flat umbilical ones, and become planes if point defects have no influence on the internal length measurement (the case of the Lie algebra $\mathbf{g}_0$). Moreover, the slip surfaces are invariant under deformations of an ideal crystal lattice characterizing the influence of a single glide on this lattice. Consequently, these slip surfaces can be considered, for dislocation lines located on them, as glide surfaces, and a relation between the scalar density of dislocations and modules of local Burgers vectors of these dislocation lines can be formulated (Section 3).

The $\mathbf{g}_t$-type distribution of dislocations can be interpreted as describing a "dislocation fluid" consisting of infinitesimal edge dislocation loops (Section 4). The density of this fluid covers with the (volume) scalar density of dislocations, and can be computed based on the equation defining slip surfaces. Consequently, the field of local Burgers vector modules of dislocation loops located on glide surfaces can be computed, too.

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